An application of Seeley's method in two dimensions

1987 J. Phys. A: Math. Gen. 20 3199

(http://iopscience.iop.org/0305-4470/20/11/023)

J. Phys. A: Math. Gen. **20** (1987) 3199-3207. Printed in the UK

An application of Seeley's method in two dimensions

Danny Birmingham†‡ and Siddhartha Sen
School of Mathematics, Trinity College, Dublin, Ireland

Received 30 September 1986

Abstract. Using Seeley's method a systematic determination of $G(x, y; \omega)$ for the equation $(-\nabla_x^2+\omega^2)G(x, y; \omega)=\delta(x-y)$, $(x, y \in \Gamma)$, is carried out for Dirichlet and Neumann boundary conditions. The region Γ is taken to be two dimensional with an arbitrary smooth boundary $\partial\Gamma$. An asymptotic expansion of $\operatorname{Tr} G^{\mathrm{B}}$ for large ω is obtained, where G^{B} is the boundary contribution to G. Using these results an earlier disagreement between Durhuus *et al* and McKean and Singer is resolved and errors in certain coefficients of $\operatorname{Tr} G^{\mathrm{B}}$ obtained by Pleijel are noted.

1. Introduction

In order to calculate determinants in string theory Durhuus *et al* (1982) had to study the operator e^{-At} where A was a second-order elliptic differential operator in two dimensions. Using an elegant method due to Seeley (1969) they obtained expressions for the kernel of e^{-At} for both Dirichlet and Neumann boundary conditions. However, the results obtained were in disagreement with previous results of McKean and Singer (1967). An essential step in the application of Seeley's method is to ensure that the space on which A is defined is the upper half-plane such that the boundary is a straight line.

In this paper we show that by a careful application of Seeley's method where we first map the two-dimensional region Γ with arbitrary boundary $\partial\Gamma$ into the upper half-plane such that the boundary is a straight line we obtain results which agree with McKean and Singer. This resolves the above disagreement which arose because sufficient attention was not paid to the geometry of the problem before Seeley's method was applied. Smith (1981) also studied the two-dimensional Dirichlet problem and developed a method for calculating $\operatorname{Tr} \mathrm{e}^{-At}$. Our results agree with those of Smith.

In our application of Seeley's method we obtain an expression for the Green function $G(x, y; \omega)$ of the Helmholtz operator $(-\nabla_x^2+\omega^2)$ for both Dirichlet and Neumann boundary conditions. We then obtain an asymptotic expansion of $\operatorname{Tr} G^{\mathrm{B}}$ for large ω, where G^{B} is the boundary contribution to G. $\operatorname{Tr} G^{\mathrm{B}}$ has been calculated using a different approach by Pleijel (1954). The results we obtain for the Dirichlet case agree with those of Pleijel except for the sign of the third term. This sign error of Pleijel was noted by Stewartson and Waechter (1971) who studied the Dirichlet problem for a region Γ with circular boundary $\partial\Gamma$ and our results confirm this as well. The results we obtain for the Neumann case agree with those of Pleijel except for the coefficient of the third term.

† Supported by an Irish Department of Education Postgraduate Award.

‡ Present address: The Blackett Laboratory, Imperial College of Science and Technology, Prince Consort Road, London SW7 2BZ, UK.

The layout of the paper is as follows. In § 2 we discuss the geometry of the problem and show how the two-dimensional region Γ with boundary $\partial\Gamma$ can be mapped into the half-plane (as far as boundary effects are concerned). In § 3 we determine $G(\boldsymbol{x}, \boldsymbol{y}; \omega)$ for both Dirichlet and Neumann boundary conditions and Tr G^{B} is computed. We also present our conclusions in § 3.

2. The geometry of the problem

In this section we consider the geometry of the problem which is an essential step before applying the method of Seeley. A readable account of Seeley's method is given by Durhuus *et al* (1982) and our treatment and notation are based on their paper. A good account of the pseudo-differential calculus is given by Gilkey (1974).

Consider a region Γ in two-dimensional Euclidean space $\mathbb{R}^2$ with an arbitrary smooth ($\equiv C^{\infty}$) boundary $\partial\Gamma$. We choose cartesian coordinates in $\mathbb{R}^2$ with $\boldsymbol{x} = (x^1, x^2) \in \mathbb{R}^2$ and flat metric δ_{ij} with the line element given by $\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2$.

We consider the coordinates in the neighbourhood of the boundary $\partial\Gamma$. Consider a strip along the interior of $\partial\Gamma$ of height $h > 0$. If h is sufficiently small we can take (s, r) as defining a local cartesian coordinate system on $\partial\Gamma$, where s is the arc length along $\partial\Gamma$ and r is the inner normal distance. The strip along $\partial\Gamma$ is then mapped into a strip along the s axis of the (s, r) plane. The boundary $\partial\Gamma$ is then given by

$$\partial\Gamma = \{(s, r) | r = 0\} \tag{2.1}$$

with

$$\Gamma = \{(s, r) | r \geqslant 0\}. \tag{2.2}$$

Having transformed coordinates from (x^1, x^2) to (s, r) we now need to find the Jacobian of this transformation. To do this we consider the diagram shown in figure 1. We have

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}x^1\, \hat{t}(s) + x^2 \hat{n}(s) \tag{2.3}$$

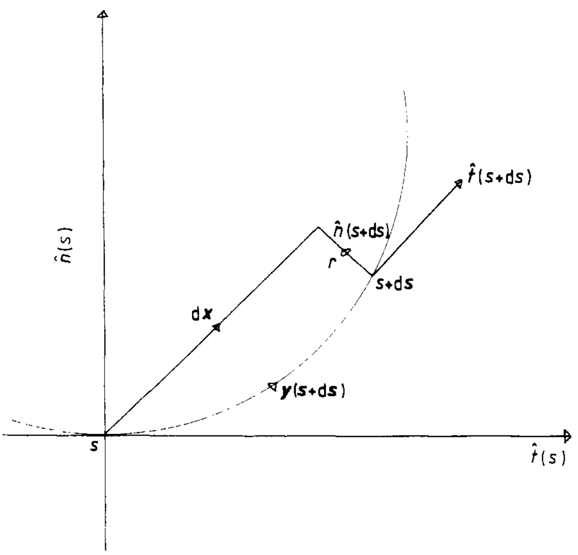

Figure 1. Transformation of coordinates in the neighbourhood of the boundary.

with respect to the origin $y(s) = \mathbf{0}$. We also have

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{y}(s+\mathrm{d}s) + r\hat{\boldsymbol{n}}(s+\mathrm{d}s) \tag{2.4}$$

with respect to the origin $\boldsymbol{y}(s+\mathrm{d}s)$. We take $\mathrm{d}s$ to be small so that we retain only $\mathrm{O}(\mathrm{d}s)$ terms. However, we note that x^2 and r need not be small. Using the Serret–Frenet formulae (Klingenberg 1978) in (2.4) we find

$$\mathrm{d}x^1 = (1 - rc(s))\,\mathrm{d}s \qquad x^2 = r \qquad \mathrm{d}x^2 = \mathrm{d}r \tag{2.5}$$

where $c(s)$ is the curvature of $\partial\Gamma$ at s. Therefore the Jacobian is given by

$$J(s, r) = (1 - rc(s)). \tag{2.6}$$

Using (2.5) we find the metric tensor in the (s, r) coordinate system is given by

$$g_{ij} = \begin{pmatrix} (1 - rc(s))^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

and the Laplacian expressed in terms of (s, r) takes the form

$$\nabla^2_{s,r} = \frac{1}{J^2}\frac{\partial^2}{\partial s^2} + \frac{rc'(s)}{J^3}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial r^2} - \frac{c(s)}{J}\frac{\partial}{\partial r}. \tag{2.8}$$

Now consider our original equation

$$(-\nabla^2_{x^1,x^2} + \omega^2)G(x^1, x^2; \omega) = \delta(x^1)\delta(x^2) \tag{2.9}$$

where we take $x = (x^1, x^2)$ and $y = (y^1, y^2) = (0, 0)$. We need to see how this equation is affected by the transformation $(x^1, x^2) \to (s, r)$. First, the delta function transforms as

$$\delta(x^1)\delta(x^2) = \frac{\delta(s)\delta(r)}{J(s, r)}. \tag{2.10}$$

Let

$$G(x^1, x^2; \omega) = G(x^1(s, r), x^2(s, r); \omega) \equiv \tilde{G}(s, r; \omega). \tag{2.11}$$

Then the transformation of (2.9) is given by

$$(-\nabla^2_{s,r} + \omega^2)\tilde{G}(s, r; \omega) = \frac{\delta(s)\delta(r)}{J(s, r)} \tag{2.12}$$

where $(-\nabla^2_{s,r})$ is given by (2.8). We define

$$\tilde{G}(s, r; \omega) \equiv \frac{H(s, r; \omega)}{J(s, r)} \tag{2.13}$$

so that we can write (2.12) as

$$(-J\nabla^2_{s,r}J^{-1} + \omega^2)H(s, r; \omega) = \delta(s)\delta(r). \tag{2.14}$$

The reason we carry out the above steps is that we want to regard $(-J\nabla^2_{s,r}J^{-1} + \omega^2)$ as an operator depending on the parameter ω and then construct the symbol of the inverse operator, denoted by $\sigma[(-J\nabla^2_{s,r}J^{-1} + \omega^2)^{-1}]$, according to the method of Seeley.

Using (2.6) and (2.8) we obtain

$$(-J\nabla^2_{s,r}J^{-1}) = \left(-\frac{1}{J^2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2}\right) + \left(-\frac{3rc'(s)}{J^3}\frac{\partial}{\partial s} - \frac{c(s)}{J}\frac{\partial}{\partial r}\right)$$
$$+\left(-\frac{c^2(s)}{J^2} - \frac{rc''(s)}{J^3} - \frac{3r^2(c'(s))^2}{J^4}\right). \tag{2.15}$$

We can then write the symbol of the operator $(-J\nabla^2_{s,r}J^{-1} + \omega^2)$ as

$$\sigma(-J\nabla^2_{s,r}J^{-1} + \omega^2) = \sum_{j=0}^{2} a_j(s, r, \xi, \tau, \omega) \tag{2.16}$$

where

$$a_2(s, r, \xi, \tau, \omega) = (G_{11}\xi^2 + \tau^2 + \omega^2) \tag{2.17}$$

$$a_0(s, r, \xi, \tau, \omega) = (G_1\xi + G_2\tau) \tag{2.18}$$

$$a_0(s, r, \xi, \tau, \omega) = G_0 \tag{2.19}$$

with

$$G_{11}(s, r) = \frac{1}{(1 - rc(s))^2} \qquad G_{22}(s, r) \equiv 1 \qquad G_1(s, r) = \frac{-3irc'(s)}{(1 - rc(s))^3}$$
$$G_2(s, r) = \frac{-ic(s)}{(1 - rc(s))} \qquad G_0(s, r) = \frac{-c^2(s)}{(1 - rc(s))^2} - \frac{rc''(s)}{(1 - rc(s))^3} - \frac{3r^2(c'(s))^2}{(1 - rc(s))^4} \tag{2.20}$$

and

$$\sigma\left(\frac{1}{i}\frac{\partial}{\partial s}\right) = \xi \qquad \sigma\left(\frac{1}{i}\frac{\partial}{\partial r}\right) = \tau. \tag{2.21}$$

We now consider the boundary conditions of the problem in terms of the new variables (s, r). Since the boundary is now given by (2.1) we have

$$\tilde{G}(s, r; \omega) = \frac{H(s, r; \omega)}{J(s, r)} = 0 \qquad \text{on } r = 0 \text{ (Dirichlet)} \tag{2.22}$$

and

$$D_r\tilde{G}(s, r; \omega) = D_r\left(\frac{H(s, r; \omega)}{J(s, r)}\right) = 0 \qquad \text{on } r = 0 \text{ (Neumann).} \tag{2.23}$$

We write the Green function as a separation

$$H(s, r; \omega) = H^0(s, r; \omega) - H^B(s, r; \omega) \tag{2.24}$$

where $H^0(s, r; \omega)$ is the Green function for the infinite plane when no boundary conditions are imposed and $H^B(s, r; \omega)$ is the compensating part due to the presence of the boundary $\partial\Gamma$ and is defined on the half-plane (2.2). The boundary conditions then take the following form:

$$BH^B(s, r; \omega) = BH^0(s, r; \omega) \qquad \text{on } r = 0 \tag{2.25}$$

with

$$B = I \qquad \text{(Dirichlet)} \tag{2.26}$$

$$B = \left(D_r + \frac{c(s)}{\mathrm{i}} I\right) \qquad \text{(Neumann)}. \tag{2.27}$$

The remaining boundary condition is

$$H^{\mathrm{B}}(s, r; \omega) \to 0 \qquad \text{as } r \to \infty. \tag{2.28}$$

Since we anticipate that $H^{\mathrm{B}}(s, r; \omega)$ will be significant only near the boundary (i.e. close to $r = 0$) we look for solutions which fall off as we move away from $\partial\Gamma$. The boundary condition (2.28) is just a statement of this fact.

Once we have constructed the symbol of the inverse operator $(-J\nabla^2_{s,r}J^{-1} + \omega^2)^{-1}$ according to the method of Seeley we can write down expressions for the Green functions as follows (Durhuus *et al* 1982):

$$\begin{aligned} H(s, r; s', r'; \omega) &= H^0(s, r; s', r'; \omega) - H^{\mathrm{B}}(s, r; s', r'; \omega) \\ &= (2\pi)^{-2} \int \exp[\mathrm{i}(s - s')\xi] \exp[\mathrm{i}(r - r')\tau] \sum_{j=0}^{\infty} c_{-2-j}(s, r, \xi, \tau, \omega)\, \mathrm{d}\xi\, \mathrm{d}\tau \\ &\quad - (2\pi)^{-2} \int \exp[\mathrm{i}(s - s')\xi] \exp(-\mathrm{i}r'\tau) \sum_{j=0}^{\infty} d_{-2-j}(s, r, \xi, \tau, \omega)\, \mathrm{d}\xi\, \mathrm{d}\tau \end{aligned} \tag{2.29}$$

where c_{-2-j} and d_{-2-j} $(j = 0, 1, 2, \ldots)$ are symbols defined by Durhuus *et al* (1982). We note the important point here is the absence of a factor $\mathrm{e}^{\mathrm{i}r\tau}$ in the expression for H^{B}. This arises because when we construct the symbols d_{-2-j} we Fourier transform in the s direction only which leads to differential equations for d_{-2-j}. When we solve these equations we find that the d_{-2-j} obtain their τ dependence solely from the boundary condition at $r = 0$. Then when we define the operator D_{-2-j} corresponding to the symbol d_{-2-j} we take the Fourier transform at $r = 0$ (Durhuus *et al* 1982, Seeley 1969). We also note that at each stage of the approximate determination of H given by (2.29) the boundary conditions are exactly satisfied.

Using (2.5), (2.13) and (2.29) we obtain the following expression for Tr G^{B} defined by

$$\begin{aligned} \operatorname{Tr} G^{\mathrm{B}} &\equiv \int_\Gamma \mathrm{d}^2 x\, G^{\mathrm{B}}(\boldsymbol{x}, \boldsymbol{x}; \omega) = \int_\Gamma \mathrm{d}s\, \mathrm{d}r\, H^{\mathrm{B}}(s, r; \omega) \\ &= (2\pi)^{-2} \int_\Gamma \mathrm{d}s\, \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\tau \exp(-\mathrm{i}r\tau) \sum_{j=0}^{\infty} d_{-2-j}(s, r, \xi, \tau, \omega) \end{aligned} \tag{2.30}$$

where

$$H^{\mathrm{B}}(s, r; \omega) \equiv H^{\mathrm{B}}(s, r; s, r; \omega). \tag{2.31}$$

We see therefore that the objects which we need to find are the symbols c_{-2-j} and d_{-2-j}. The form of these symbols depends on the operator which we are studying, namely $(-J\nabla^2_{s,r}J^{-1} + \omega^2)$, and also on the boundary conditions.

3. Results and conclusions

In this section we determine the first three terms in the series expansion for the Green function H defined by (2.29) for both Dirichlet and Neumann boundary conditions. We then use the symbols d_{-2-j} $(j=0,1,2)$ to evaluate $\operatorname{Tr} G^{\mathrm{B}}$ given by (2.30) as an asymptotic expansion for large ω. Our conclusions are then presented.

The Green function $H^0(s, r; s', r'; \omega)$ given by (2.29) is defined via the symbols c_{-2-j} for which we obtain the following results:

$$c_{-2}=(\tau^2+\Lambda^2)^{-1} \tag{3.1}$$

where

$$\Lambda^2 \equiv (G_{11}\xi^2+\omega^2) \tag{3.2}$$

$$c_{-3}=-\frac{(G_1\xi+G_2\tau)}{(\tau^2+\Lambda^2)^2}+\frac{(-\mathrm{i}2G_{11}\xi)(G_{11,1}\xi^2)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2\tau)(G_{11,2}\xi^2)}{(\tau^2+\Lambda^2)^3} \tag{3.3}$$

$$\begin{aligned}
c_{-4}=&-\frac{1}{(\tau^2+\Lambda^2)}\left[\frac{G_0}{(\tau^2+\Lambda^2)}+(-\mathrm{i}G_1)\frac{(-G_{11,1}\xi^2)}{(\tau^2+\Lambda^2)^2}\right.\\
&+(-\mathrm{i}G_2)\frac{(-G_{11,2}\xi^2)}{(\tau^2+\Lambda^2)^2}+(-G_{11})\left(\frac{2(G_{11,1}\xi^2)^2}{(\tau^2+\Lambda^2)^3}-\frac{G_{11,1,1}\xi^2}{(\tau^2+\Lambda^2)^2}\right)\\
&-\left(\frac{2(G_{11,2}\xi^2)^2}{(\tau^2+\Lambda^2)^3}-\frac{G_{11,2,2}\xi^2}{(\tau^2+\Lambda^2)^2}\right)-\frac{(G_1\xi+G_2\tau)^2}{(\tau^2+\Lambda^2)^2}\\
&+\frac{(-\mathrm{i}2G_{11}\xi)(G_{11,1}\xi^2)(G_1\xi+G_2\tau)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2\tau)(G_{11,2}\xi^2)(G_1\xi+G_2\tau)}{(\tau^2+\Lambda^2)^3}\\
&-\frac{(-\mathrm{i}2G_{11}\xi)(G_{1,1}\xi+G_{2,1}\tau)}{(\tau^2+\Lambda^2)^2}+2\frac{(-\mathrm{i}2G_{11}\xi)(G_1\xi+G_2\tau)(G_{11,1}\xi^2)}{(\tau^2+\Lambda^2)^3}\\
&+\frac{(-\mathrm{i}2G_{11}\xi)(-\mathrm{i}2G_{11,1}\xi)(G_{11,1}\xi^2)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2G_{11}\xi)^2(G_{11,1,1}\xi^2)}{(\tau^2+\Lambda^2)^3}\\
&+\frac{(-\mathrm{i}2G_{11}\xi)^2(G_{11,1}\xi^2)^2(-3)}{(\tau^2+\Lambda^2)^4}+\frac{(-\mathrm{i}2G_{11}\xi)(-\mathrm{i}2\tau)(G_{11,2,1}\xi^2)}{(\tau^2+\Lambda^2)^3}\\
&+\frac{(-\mathrm{i}2G_{11}\xi)(-\mathrm{i}2\tau)(G_{11,2}\xi^2)(G_{11,1}\xi^2)(-3)}{(\tau^2+\Lambda^2)^4}-\frac{(-\mathrm{i}2\tau)(G_{1,2}\xi+G_{2,2}\tau)}{(\tau^2+\Lambda^2)^2}\\
&+\frac{(-\mathrm{i}2\tau)2(G_1\xi+G_2\tau)(G_{11,2}\xi^2)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2\tau)(-\mathrm{i}2G_{11,2}\xi)(G_{11,1}\xi^2)}{(\tau^2+\Lambda^2)^3}\\
&+\frac{(-\mathrm{i}2\tau)(-\mathrm{i}2G_{11}\xi)(G_{11,1,2}\xi^2)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2\tau)(-\mathrm{i}2G_{11}\xi)(G_{11,1}\xi^2)(G_{11,2}\xi^2)(-3)}{(\tau^2+\Lambda^2)^4}\\
&\left.+\frac{(-\mathrm{i}2\tau)^2(G_{11,2,2}\xi^2)}{(\tau^2+\Lambda^2)^3}+\frac{(-\mathrm{i}2\tau)^2(G_{11,2}\xi^2)^2(-3)}{(\tau^2+\Lambda^2)^4}\right].
\end{aligned} \tag{3.4}$$

The Green function $H^{\mathrm{B}}(s, r; s', r'; \omega)$ given by (2.29) is defined via the symbols d_{-2-j} for which we obtain the following results.

The Dirichlet case

$$d_{-2}=\frac{\exp(-r\Lambda_0)}{(\tau^2+\Lambda_0^2)} \tag{3.5}$$

where

$$\Lambda_0^2 \equiv \Lambda^2(r=0) = (\xi^2+\omega^2) \tag{3.6}$$

$$d_{-3} = \exp(-r\Lambda_0)\left(\frac{-4i\tau\xi^2 c(s)}{(\tau^2+\Lambda_0^2)^3} + \frac{i\tau c(s)}{(\tau^2+\Lambda_0^2)^2}\right) + r\exp(-r\Lambda_0)\left(\frac{-c(s)\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)} - \frac{c(s)}{2(\tau^2+\Lambda_0^2)}\right) + r^2\exp(-r\Lambda_0)\left(\frac{-c(s)\xi^2}{2\Lambda_0(\tau^2+\Lambda_0^2)}\right) \tag{3.7}$$

$$\begin{aligned} d_{-4} = {} & \exp(-r\Lambda_0)\left(\frac{c^2(s)}{(\tau^2+\Lambda_0^2)^2} - \frac{8c^2(s)\xi^2}{(\tau^2+\Lambda_0^2)^3} + \frac{8c^2(s)\xi^4}{(\tau^2+\Lambda_0^2)^4} - \frac{3c^2(s)\tau^2}{(\tau^2+\Lambda_0^2)^3} + \frac{36c^2(s)\tau^2\xi^2}{(\tau^2+\Lambda_0^2)^4}\right. \\ & \left. - \frac{48c^2(s)\tau^2\xi^4}{(\tau^2+\Lambda_0^2)^5} - \frac{8c'(s)\tau\xi}{(\tau^2+\Lambda_0^2)^3} + \frac{16c'(s)\tau\xi^3}{(\tau^2+\Lambda_0^2)^4}\right) \\ & + r\exp(-r\Lambda_0)\left(\frac{5c^2(s)\xi^4}{8\Lambda_0^5(\tau^2+\Lambda_0^2)} - \frac{3c^2(s)\xi^2}{4\Lambda_0^3(\tau^2+\Lambda_0^2)} + \frac{c^2(s)}{8\Lambda_0(\tau^2+\Lambda_0^2)} + \frac{2ic^2(s)\tau\xi^4}{\Lambda_0^2(\tau^2+\Lambda_0^2)^3}\right. \\ & - \frac{ic^2(s)\tau\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)^2} + \frac{2ic^2(s)\tau\xi^2}{(\tau^2+\Lambda_0^2)^3} - \frac{ic^2(s)\tau}{2(\tau^2+\Lambda_0^2)^2} + \frac{ic'(s)\xi}{2\Lambda_0^2(\tau^2+\Lambda_0^2)} \\ & \left. - \frac{ic'(s)\xi^3}{2\Lambda_0^4(\tau^2+\Lambda_0^2)} + \frac{4c'(s)\tau\xi^3}{\Lambda_0(\tau^2+\Lambda_0^2)^3} - \frac{c'(s)\tau\xi}{\Lambda_0(\tau^2+\Lambda_0^2)^2}\right) \\ & + r^2\exp(-r\Lambda_0)\left(\frac{5c^2(s)\xi^4}{8\Lambda_0^4(\tau^2+\Lambda_0^2)} - \frac{c^2(s)\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)} - \frac{c^2(s)}{8(\tau^2+\Lambda_0^2)} + \frac{2ic^2(s)\tau\xi^4}{\Lambda_0(\tau^2+\Lambda_0^2)^3}\right. \\ & \left. - \frac{ic^2(s)\tau\xi^2}{2\Lambda_0(\tau^2+\Lambda_0^2)^2} + \frac{ic'(s)\xi}{2\Lambda_0(\tau^2+\Lambda_0^2)} - - \frac{ic'(s)\xi^3}{2\Lambda_0^3(\tau^2+\Lambda_0^2)}\right) \\ & + r^3\exp(-r\Lambda_0)\left(\frac{5c^2(s)\xi^4}{12\Lambda_0^3(\tau^2+\Lambda_0^2)} - \frac{c^2(s)\xi^2}{4\Lambda_0(\tau^2+\Lambda_0^2)} - \frac{ic'(s)\xi^3}{6\Lambda_0^2(\tau^2+\Lambda_0^2)}\right) \\ & + r^4\exp(-r\Lambda_0)\left(\frac{c^2(s)\xi^4}{8\Lambda_0^2(\tau^2+\Lambda_0^2)}\right). \end{aligned} \tag{3.8}$$

The Neumann case

$$d_{-2} = -\frac{i\tau\exp(-r\Lambda_0)}{\Lambda_0(\tau^2+\Lambda_0^2)} \tag{3.9}$$

$$\begin{aligned} d_{-3} = {} & \exp(-r\Lambda_0)\left(\frac{i\tau c(s)\xi^2}{2\Lambda_0^4(\tau^2+\Lambda_0^2)} + \frac{c(s)i\tau}{2\Lambda_0^2(\tau^2+\Lambda_0^2)} - \frac{c(s)}{\Lambda_0(\tau^2+\Lambda_0^2)} - \frac{i\tau c(s)}{\Lambda_0^2(\tau^2+\Lambda_0^2)}\right. \\ & \left. - \frac{4\tau^2 c(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^3} + \frac{\tau^2 c(s)}{\Lambda_0(\tau^2+\Lambda_0^2)^2} + \frac{2c(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^2}\right) \\ & + r\exp(-r\Lambda_0)\left(\frac{i\tau c(s)\xi^2}{2\Lambda_0^3(\tau^2+\Lambda_0^2)} + \frac{i\tau c(s)}{2\Lambda_0(\tau^2+\Lambda_0^2)}\right) \\ & + r^2\exp(-r\Lambda_0)\left(\frac{i\tau c(s)\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)}\right) \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
d_{-4} = \exp(-r\Lambda_0)\bigg(& -\frac{7i\tau c^2(s)\xi^4}{8\Lambda_0^7(\tau^2+\Lambda_0^2)}+\frac{5i\tau c^2(s)\xi^2}{4\Lambda_0^5(\tau^2+\Lambda_0^2)}-\frac{3i\tau c^2(s)}{8\Lambda_0^3(\tau^2+\Lambda_0^2)}-\frac{c^2(s)\xi^4}{\Lambda_0^4(\tau^2+\Lambda_0^2)^2} \\
& +\frac{2\tau^2c^2(s)\xi^4}{\Lambda_0^4(\tau^2+\Lambda_0^2)^3}-\frac{\tau^2c^2(s)\xi^2}{2\Lambda_0^4(\tau^2+\Lambda_0^2)^2}+\frac{c^2(s)\xi^2}{2\Lambda_0^4(\tau^2+\Lambda_0^2)}+\frac{c^2(s)\xi^2}{\Lambda_0^2(\tau^2+\Lambda_0^2)^2} \\
& -\frac{2\tau^2c^2(s)\xi^2}{\Lambda_0^2(\tau^2+\Lambda_0^2)^3}+\frac{\tau^2c^2(s)}{2\Lambda_0^2(\tau^2+\Lambda_0^2)^2}-\frac{c^2(s)}{2\Lambda_0^2(\tau^2+\Lambda_0^2)}+\frac{28i\tau c^2(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^3} \\
& -\frac{3i\tau c^2(s)}{\Lambda_0(\tau^2+\Lambda_0^2)^2}-\frac{32i\tau c^2(s)\xi^4}{\Lambda_0(\tau^2+\Lambda_0^2)^4} \\
& +\frac{3i\tau^3c^2(s)}{\Lambda_0(\tau^2+\Lambda_0^2)^3}-\frac{36i\tau^3c^2(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^4}+\frac{48i\tau^3c^2(s)\xi^4}{\Lambda_0(\tau^2+\Lambda_0^2)^5} \\
& +\frac{c'(s)\tau\xi}{\Lambda_0^4(\tau^2+\Lambda_0^2)}-\frac{\tau c'(s)\xi^3}{\Lambda_0^6(\tau^2+\Lambda_0^2)}-\frac{ic'(s)\xi}{\Lambda_0^3(\tau^2+\Lambda_0^2)}-\frac{4i\tau^2c'(s)\xi^3}{\Lambda_0^2(\tau^2+\Lambda_0^2)^3}+\frac{i\tau^2c'(s)\xi}{\Lambda_0^3(\tau^2+\Lambda_0^2)^2} \\
& +\frac{2ic'(s)\xi^3}{\Lambda_0^3(\tau^2+\Lambda_0^2)^2}-\frac{3ic'(s)\xi}{\Lambda_0(\tau^2+\Lambda_0^2)^2}+\frac{4ic'(s)\xi^3}{\Lambda_0(\tau^2+\Lambda_0^2)^3}+\frac{8i\tau^2c'(s)\xi}{\Lambda_0(\tau^2+\Lambda_0^2)^3} \\
& -\frac{16i\tau^2c'(s)\xi^3}{\Lambda_0(\tau^2+\Lambda_0^2)^4}\bigg)+r\exp(-r\Lambda_0)\bigg(-\frac{7i\tau c^2(s)\xi^4}{8\Lambda_0^6(\tau^2+\Lambda_0^2)}+\frac{3i\tau c^2(s)\xi^2}{4\Lambda_0^4(\tau^2+\Lambda_0^2)} \\
& +\frac{i\tau c^2(s)}{8\Lambda_0^2(\tau^2+\Lambda_0^2)}-\frac{c^2(s)\xi^4}{\Lambda_0^3(\tau^2+\Lambda_0^2)^2}+\frac{2\tau^2c^2(s)\xi^4}{\Lambda_0^3(\tau^2+\Lambda_0^2)^3}-\frac{\tau^2c^2(s)\xi^2}{2\Lambda_0^3(\tau^2+\Lambda_0^2)^2} \\
& +\frac{c^2(s)\xi^2}{2\Lambda_0^3(\tau^2+\Lambda_0^2)}-\frac{c^2(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^2}+\frac{2\tau^2c^2(s)\xi^2}{\Lambda_0(\tau^2+\Lambda_0^2)^3} \\
& -\frac{\tau^2c^2(s)}{2\Lambda_0(\tau^2+\Lambda_0^2)^2}+\frac{c^2(s)}{2\Lambda_0(\tau^2+\Lambda_0^2)}+\frac{c'(s)\tau\xi}{\Lambda_0^3(\tau^2+\Lambda_0^2)}-\frac{\tau c'(s)\xi^3}{\Lambda_0^5(\tau^2+\Lambda_0^2)}-\frac{ic'(s)\xi}{\Lambda_0^2(\tau^2+\Lambda_0^2)} \\
& -\frac{4i\tau^2c'(s)\xi^3}{\Lambda_0^2(\tau^2+\Lambda_0^2)^3}+\frac{i\tau^2c'(s)\xi}{\Lambda_0^2(\tau^2+\Lambda_0^2)^2}+\frac{2ic'(s)\xi^3}{\Lambda_0^2(\tau^2+\Lambda_0^2)^2}\bigg) \\
& +r^2\exp(-r\Lambda_0)\bigg(-\frac{7i\tau c^2(s)\xi^4}{8\Lambda_0^5(\tau^2+\Lambda_0^2)}+\frac{3i\tau c^2(s)\xi^2}{4\Lambda_0^3(\tau^2+\Lambda_0^2)}+\frac{i\tau c^2(s)}{8\Lambda_0(\tau^2+\Lambda_0^2)} \\
& -\frac{c^2(s)\xi^4}{\Lambda_0^2(\tau^2+\Lambda_0^2)^2}+\frac{2\tau^2c^2(s)\xi^4}{\Lambda_0^2(\tau^2+\Lambda_0^2)^3}-\frac{\tau^2c^2(s)\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)^2}+\frac{c^2(s)\xi^2}{2\Lambda_0^2(\tau^2+\Lambda_0^2)} \\
& +\frac{c'(s)\tau\xi}{2\Lambda_0^2(\tau^2+\Lambda_0^2)}-\frac{\tau c'(s)\xi^3}{2\Lambda_0^4(\tau^2+\Lambda_0^2)}\bigg) \\
& +r^3\exp(-r\Lambda_0)\bigg(\frac{-5i\tau c^2(s)\xi^4}{12\Lambda_0^4(\tau^2+\Lambda_0^2)}+\frac{i\tau c^2(s)\xi^2}{4\Lambda_0^2(\tau^2+\Lambda_0^2)}-\frac{c'(s)\tau\xi^3}{6\Lambda_0^3(\tau^2+\Lambda_0^2)}\bigg) \\
& +r^4\exp(-r\Lambda_0)\bigg(\frac{-i\tau c^2(s)\xi^4}{8\Lambda_0^3(\tau^2+\Lambda_0^2)}\bigg).
\end{aligned} \tag{3.11}
$$

Using the symbols d_{-2-j} $(j=0,1,2)$ we can now evaluate the quantity Tr G^{B} given by (2.30) as an asymptotic expansion in powers of (ω^{-1}) for large ω. The order of integration which we adopt is (τ, r, ξ, s). We note here that the terms in d_{-4} Dirichlet and d_{-4} Neumann which are proportional to $c'(s)$ are also odd functions of ξ. Since we need to perform a ξ integration over the range $(-\infty, \infty)$ we can neglect these terms when evaluating Tr G^{B}. We obtain the following results:

$$\operatorname{Tr} G^{\mathrm{B}}=\frac{S}{8\omega}-\frac{1}{12\pi\omega^2}\int_{\partial\Gamma} c(s)\,\mathrm{d}s-\frac{1}{512\omega^3}\int_{\partial\Gamma} c^2(s)\,\mathrm{d}s+\mathrm{O}(\omega^{-4}) \qquad \text{(Dirichlet)} \tag{3.12}$$

$$\operatorname{Tr} G^{\mathrm{B}}=\frac{-S}{8\omega}-\frac{1}{12\pi\omega^2}\int_{\partial\Gamma} c(s)\,\mathrm{d}s-\frac{5}{512\omega^3}\int_{\partial\Gamma} c^2(s)\,\mathrm{d}s+\mathrm{O}(\omega^{-4}) \qquad \text{(Neumann)} \tag{3.13}$$

where S is the length of the boundary $\partial\Gamma$. These results agree with those obtained by Pleijel (1954) except for the coefficients of the $\int_{\partial\Gamma} c^2(s)\,\mathrm{d}s$ term for the Dirichlet and Neumann problems. Pleijel obtained $(+\frac{1}{512}\omega^{-3}, -\frac{7}{512}\omega^{-3})$ as the coefficients of the Dirichlet and Neumann problems, respectively. The sign error of Pleijel in the Dirichlet case was noted by Stewartson and Waechter (1971) who performed the calculation for a region Γ with circular boundary $\partial\Gamma$. Therefore our result agrees with that of Stewartson and Waechter.

Using the asymptotic expansions (3.12) and (3.13) it is possible to obtain asymptotic expansions of $\operatorname{Tr} \mathrm{e}^{-At}$ as $t \to 0^+$ where A is the Laplacian $(-\nabla^2)$ (Stewartson and Waechter 1971). The resulting expansions are then in agreement with those obtained by McKean and Singer (1967) and also by Smith (1981).

The symbols c_{-2}, c_{-3}, c_{-4}, d_{-2} and d_{-3} for both the Dirichlet and Neumann problems were constructed earlier by Durhuus *et al* (1982). Our results agree with the above except for the symbol d_{-3} in the Neumann case. The source of this disagreement can be traced to the fact that the boundary operator which we have used in the Neumann case, namely (2.27), is different from that used by Durhuus *et al.* Consequently, we have two extra terms in d_{-3}, namely $\exp(-r\Lambda_0)[-c(s)/(\Lambda_0(\tau^2+\Lambda_0^2)]$ and $\exp(-r\Lambda_0)[-\mathrm{i}\tau c(s)/\Lambda_0^2(\tau^2+\Lambda_0^2)]$. Our result for d_{-3} agrees with the expression obtained by Durhuus *et al* apart from these two terms. The reason the boundary operator used by us is different is as follows. We recall from § 2 that when we performed the coordinate transformation $(x^1, x^2) \to (s, r)$ we also had to transform our basic equation (2.9). This led us to define $\tilde{G}(s, r; \omega) \equiv H(s, r; \omega)/J(s, r)$ and consequently the Neumann boundary condition led to the boundary operator $B=(D_r-\mathrm{i}c(s)I)$. As discussed in § 2 this transformation of the problem to one involving a local cartesian coordinate system (s, r), where the boundary $\partial\Gamma$ was a straight line, was an essential step before applying the method of Seeley.

Finally it should be apparent that once $G(\boldsymbol{x}, \boldsymbol{y}; \omega)$ is determined many problems can be tackled. For instance, we can proceed to do cavity field theory and use $G(\boldsymbol{x}, \boldsymbol{y}; \omega)$ to construct propagators that can be used to write down the Feynman rules for field theory in a cavity. It is also possible to generalise the procedure to higher dimensions.

References

Durhuus B, Olesen P and Petersen J L 1982 *Nucl. Phys.* B **198** 157

Gilkey P B 1974 *The Index Theorem and the Heat Equation* (Boston, MA: Publish or Perish)

Klingenberg W 1978 *A Course in Differential Geometry* (Berlin: Springer)

McKean H P and Singer I M 1967 *J. Diff. Geom.* **1** 43

Pleijel A 1954 *Ark. Matematik* **2** 553

Seeley R T 1969 *Am. J. Math.* **91** 889, 963

Smith L 1981 *Invent. Math.* **63** 467

Stewartson K and Waechter R T 1971 *Proc. Camb. Phil. Soc.* **69** 353